Constructing a regular n-simplex

We will construct a regular n-simplex in $\mathbb{R}^n$.
To this end, take $e_1, e_2, \ldots, e_{n+1} \in \mathbb{R}^{n+1}$. Now, we wish to choose an orthonormal affine basis of the affine subspace generated by these vectors. So, first of all, we must translate the affine space generated by these vectors so to include 0. We have
$\mathrm{aff}(e_1, e_2, \ldots, e_{n+1}) = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1\}$.
So, we will simply take

$$y = \begin{bmatrix} -\frac{1}{n+1} \\ -\frac{1}{n+1} \\ \vdots \\ -\frac{1}{n+1} \end{bmatrix} \in \mathbb{R}^{n+1}$$

and consider the vectors $f_1 = e_1 - y, f_2 = e_2 - y, \ldots, f_{n+1} = e_{n+1} - y$. Now, we have
$\mathrm{aff}(f_1, f_2, \ldots, f_{n+1}) = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\} = U \subseteq \mathbb{R}^{n+1}$.
Now, we must choose an orthonormal basis of the vector space U. We will choose the following basis:

$$g_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, g_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, g_3 = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, g_n = \frac{1}{\sqrt{n+n^2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ -n \end{bmatrix} \in U$$

Next, we must calculate a matrix $A \in \mathbb{R}^{n \times (n+1)}$ that takes a vector from U and gives us the corresponding coordinates in terms of the basis $g_1, g_2, \ldots, g_n$. Equivalently, take $B \in \mathbb{R}^{(n+1) \times n}$ with the i-th column of B being g_i. This means that B is the matrix that takes a vector of coordinates in terms of the basis $g_1, g_2, \ldots, g_n$ and gives us a vector from U. Then we simply want to find a matrix A with $AB = \mathrm{id}_n \in \mathbb{R}^{n \times n}$.
So, we have:

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{n+n^2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{n+n^2}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{n+n^2}} \\ 0 & 0 & -\frac{3}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{n+n^2}} \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{n+n^2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{n+n^2}} \\ 0 & 0 & 0 & \cdots & -\frac{n}{\sqrt{n+n^2}} \end{bmatrix} \in \mathbb{R}^{(n+1) \times n}$$

Now, we can see that we can simply take $A = B^T$, essentially since the basis we chose was orthonormal. Hence we obtain the following coordinates for our regular n-simplex in $\mathbb{R}^n$:

$$Af_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{20}} \\ \vdots \\ \frac{1}{\sqrt{n+n^2}} \end{bmatrix}, Af_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{20}} \\ \vdots \\ \frac{1}{\sqrt{n+n^2}} \end{bmatrix}, Af_3 = \begin{bmatrix} 0 \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{20}} \\ \vdots \\ \frac{1}{\sqrt{n+n^2}} \end{bmatrix}, Af_4 = \begin{bmatrix} 0 \\ 0 \\ -\frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{20}} \\ \vdots \\ \frac{1}{\sqrt{n+n^2}} \end{bmatrix}, Af_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{4}{\sqrt{20}} \\ \vdots \\ \frac{1}{\sqrt{n+n^2}} \end{bmatrix}, \ldots, Af_{n+1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -\frac{n}{\sqrt{n+n^2}} \end{bmatrix} \in \mathbb{R}^n$$

For example, for $n = 2$ we obtain the following 2-simplex in $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{\sqrt{6}} \end{bmatrix} \right\} \subseteq \mathbb{R}^2$$

And for $n = 3$ we obtain the following 3-simplex in $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{12}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\frac{3}{\sqrt{12}} \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$